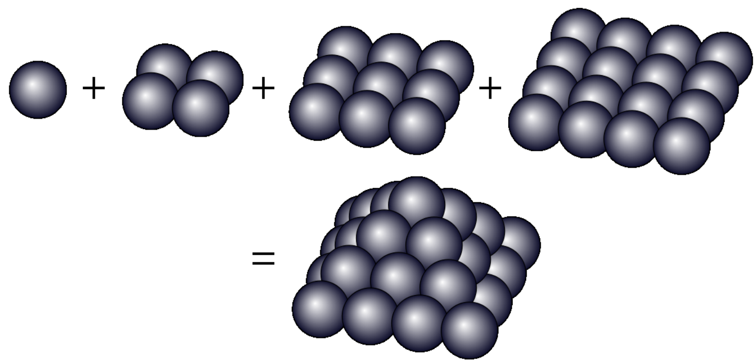
Square pyramidal number

In mathematics, a **pyramid number**, or **square pyramidal number**, is a figurate number that represents the number of stacked spheres in a pyramid with a square base. Square pyramidal numbers also solve the problem of counting the number of squares in an $n \times n$ grid.



Geometric representation of the square pyramidal number 1 + 4 + 9 + 16 = 30.

Formula

The first few square pyramidal numbers are:

1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819 (sequence A000330 in OEIS).

These numbers can be expressed in a formula as

$$P_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6}.$$

This is a special case of Faulhaber's formula, and may be proved by a straightforward mathematical induction. An equivalent formula is given in Fibonacci's Liber Abaci (1202, ch. II.12).

In modern mathematics, figurate numbers are formalized by the Ehrhart polynomials. The Ehrhart polynomial $L(P,t)$ of a polyhedron P is a polynomial that counts the number of integer points in a copy of P that is expanded by multiplying all its coordinates by the number t. The Ehrhart polynomial of a pyramid whose base is a unit square with integer coordinates, and whose apex is an integer point at height one above the base plane, is $(t + 1)(t + 2)(2t + 3)/6 = P_{t+1}$.[1]

Relations to other figurate numbers

The square pyramidal numbers can also be expressed as sums of binomial coefficients:

$$P_n = \binom{n+2}{3} + \binom{n+1}{3}.$$

The binomial coefficients occurring in this representation are tetrahedral numbers, and this formula expresses a square pyramidal number as the sum of two tetrahedral numbers in the same way as square numbers are the sums of two consecutive triangular numbers. In this sum, one of the two tetrahedral numbers counts the number of balls in a stacked pyramid that are directly above or to one side of a diagonal of the base square, and the other tetrahedral number in the sum counts the number of balls that are to the other side of the diagonal. Square pyramidal numbers are also related to tetrahedral numbers in a different way:

$$P_n = \frac{1}{4}\binom{2n+2}{3}.$$

The sum of two consecutive square pyramidal numbers is an octahedral number.

Augmenting a pyramid whose base edge has n balls by adding to one of its triangular faces a tetrahedron whose base edge has $n - 1$ balls produces a triangular prism. Equivalently, a pyramid can be expressed as the result of subtracting a tetrahedron from a prism. This geometric dissection leads to another relation:

$$P_n = n\binom{n+1}{2} - \binom{n+1}{3}.$$

Besides 1, there is only one other number that is both a square and a pyramid number: 4900, which is both the 70th square number and the 24th square pyramidal number. This fact was proven by G. N. Watson in 1918.

Another relationship involves the Pascal Triangle: Whereas the classical Pascal Triangle with sides (1,1) has diagonals with the natural numbers, triangular numbers, and tetrahedral numbers, generating the Fibonacci numbers as sums of samplings across diagonals, the sister Pascal with sides (2,1) has equivalent diagonals with odd numbers, square numbers, and square pyramidal numbers, respectively, and generates (by the same procedure) the Lucas numbers rather than Fibonacci.

In the same way that the square pyramidal numbers can be defined as a sum of consecutive squares, the squared triangular numbers can be defined as a sum of consecutive cubes.

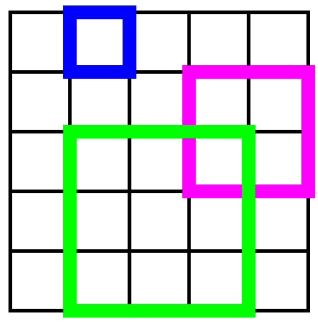
Squares in a square

A common mathematical puzzle involves finding the number of squares in a large *n* by *n* square grid. This number can be derived as follows:

- The number of 1×1 boxes found in the grid is n^2.
- The number of 2×2 boxes found in the grid is $(n-1)^2$. These can be counted by counting all of the possible upper-left corners of 2×2 boxes.
- The number of *k*×*k* boxes ($1 \le k \le n$) found in the grid is $(n-k+1)^2$. These can be counted by counting all of the possible upper-left corners of *k*×*k* boxes.

It follows that the number of squares in an *n* by *n* square grid is:

$$n^2 + (n-1)^2 + (n-2)^2 + (n-3)^2 + \ldots + 1^2 = \frac{n(n+1)(2n+1)}{6}.$$



A 5 by 5 square grid, with three of its 55 squares highlighted.

That is, the solution to the puzzle is given by the square pyramidal numbers.

The number of rectangles in a square grid is given by the squared triangular numbers.

Notes

[1] Beck, M.; De Loera, J. A.; Develin, M.; Pfeifle, J.; Stanley, R. P. (2005), "Coefficients and roots of Ehrhart polynomials", *Integer points in polyhedra—geometry, number theory, algebra, optimization*, Contemp. Math., **374**, Providence, RI: Amer. Math. Soc., pp. 15–36, MR2134759.

References

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External links

- Weisstein, Eric W., " Square Pyramidal Number (http://mathworld.wolfram.com/SquarePyramidalNumber.html)" from MathWorld.

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